Competing Mechanisms in Sequential Sales*

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Abstract

Two sellers, each with one identical good, compete sequentially to sell to $N \geq 3$ buyers. Buyers who do not get Seller 1's good subsequently participate in a second-price auction for Seller 2's good. Seller 2 chooses a reserve price r, and Seller 1 may choose any mechanism. We characterize the optimal mechanism for Seller 1 as a function of r. The first-order approach typically fails, so we develop new techniques. The optimal mechanism cannot be implemented by a standard auction; instead we present a modified third-price auction implementation. We then characterize Seller 2's optimal response and equilibrium outcomes of competition in mechanisms.

1 Introduction

Sequential auctions to sell multiple units of identical units owned by different sellers are common. On platforms like eBay, individual sellers run auctions ending at different times. Auction houses such as Sotheby's and Christie's sell goods in a sequence of

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single-object auctions on behalf of different sellers. Much of the theoretical literature on sequential auctions focuses on characterizing the equilibrium behavior of bidders under the assumption that sellers are passive and nonstrategic. In this paper, we analyze how this type of type of competition between sellers separated across time affects the optimal design of selling mechanisms.

We study a simple environment. Two sellers, each with one unit of an identical good, sell sequentially to $N \geq 3$ buyers. Buyers have unit demands, with private values independently drawn from a common distribution F with density f. A buyer who fails to obtain the first seller's good then participates in the second seller's auction. The second seller uses a second-price auction with reserve price r. The first seller can choose any selling mechanism. Restricting the second seller to a second-price auction, in which buyers have a dominant strategy to bid their value, means that we can abstract from the issue of information leakage, where buyers may learn something about each other's values from the outcome of the first mechanism. We will further assume that the second seller commits to her reserve price r before the first mechanism runs, so our analysis is similarly robust to learning by the second seller.

We start by deriving the revenue-maximizing pricing and allocation rule for the first seller as a function of r. Hendricks and Wiseman [forthcoming] calculates that optimal mechanism for the case where the second seller uses a second-price auction with no reserve. Allowing for a non-trivial reserve price r_2 introduces a significant complication: the solution to the first seller's optimization problem given first-order incentive constraints turns out to violate global incentive compatibility when r is less than $\psi^{-1}(0)$, the optimal reserve price in the single seller case. As a consequence, the standard methods from Myerson [1981] are inadequate. Instead, we characterize the solution using techniques that, like those of Bergemann et al. [2020] and Carroll and Segal [2019], may be useful in other mechanism design settings where the first-order approach fails.

The revenue-maximizing mechanism for the first seller is very different from that of a monopoly seller. A monopolist optimally allocates to the buyer with the highest reported type whenever that type is above a threshold that is independent of the number of bidders N. In our setting, the optimal allocation rule depends on the first-, second-, and third-highest reported values, and it varies with N. A standard auction with a reserve price clearly cannot implement that allocation rule. Instead, we construct a modified third-price auction that implements the optimal mechanism. Intuitively, a standard auction does not work well for the first seller because there is an allocation externality. A bidder's outside option is his endogenous payoff from competing in the second auction. Allocating the first good to a high-value bidder reduces the subsequent competition that other buyers face.[1]

Having derived that solution, we then use it to analyze a simple form of competition in mechanism between the two sellers. We model the strategic timing as follows: the second seller chooses her reserve price r, and the first seller responds with an optimal mechanism given r. We show that an equilibrium exists and that the equilibrium reserve price r is below the monopoly price $\psi^{-1}(0)$. In an example with three buyers whose valuations are distributed uniformly on the unit interval, we derive the equilibrium outcome of the strategic interaction between the sellers and find that the second seller uses a reserve price to increase her revenue at the expense of the first seller.

We formulate our model in terms of sale auctions, but we can equally interpret it as a model of procurement auctions, where the bidders are potential sellers and their types represent their production costs. An important motivating example is the market for pharmaceuticals in Ecuador and other middle-income countries, studied by Brugués [2020]. There, the first buyer is the government, who procures a supply for the public market. Losing bidders compete to serve the private market, where Bertrand competition yields the same outcome as a second-price auction with no reserve. Our analysis may be especially relevant in this environment, because a government agency may have greater ability to implement non-standard auction rules than would a private seller.

The literature on competing mechanisms has mostly focused on markets where

[1] Work on auctions with more general externalities includes Jehiel et al. [1996, 1999], Jehiel and Moldovanu [2003], and Figueroa and Skreta [2009]. Bergemann et al. [2020], Calzolari and Pavan [2006], Carroll and Segal [2019], Dworczak [2020], and Virág [2016] contribute to the recent literature on optimal design of auctions and disclosure rules when externalities result from resale.

sellers with identical goods choose their mechanisms simultaneously and buyers then select among them. Burguet and Sákovics [1999], studying the case of two sellers who simultaneously choose reserve prices in second-price auctions, find that competition for buyers lowers equilibrium reserve prices, but not to zero. McAfee [1993], Peters and Severinov [1997], and Pai [2009] consider the general mechanism choice problem and show that, when the number of sellers and buyers is large, second-price auctions with zero reserve prices emerge as an equilibrium mechanism.

Closer to our paper, Kirkegaard and Overgaard [2008] study Black and de Meza [1992]'s model of two sequential, second-price auctions when buyers have multi-unit demands. They show that the early seller can increase her expected revenue by offering an optimal buy-out price. Finally, Hendricks and Wiseman [forthcoming] derive the first seller's optimal mechanism in the special case of our model where the second seller is passive and sets no reserve price. The solution there, which is simpler to characterize because the failure of incentive compatibility under the first-order approach does not arise, preserves the basic features of the optimal mechanism for the case where r is nontrivial. Here we build on that analysis in order to make the second seller a strategic agent.

2 Model

There are $N \geq 3$ *ex ante* identical potential buyers, indexed by i, with unit demand for an indivisible good. Each buyer i's privately observed valuation for the good X_i is independently drawn from distribution F with support $[\underline{x}, \bar{x}]$, $\underline{x} \geq 0$. We will sometimes refer to a buyer's valuation as his *type*. We assume that F has a continuous density f and that the virtual valuation $\psi(x) \equiv x - (1 - F(x))/f(x)$ is increasing in x.

There are two sellers who each sell one unit of an identical good. They sell their units sequentially over two periods, and we refer to them in the order that they sell. The second seller uses a second-price auction with reserve price $r \geq \underline{x}$. Given r, the first seller chooses his mechanism. Both sellers' valuations of the good are normalized to zero. This structure is common knowledge. We will characterize the revenue-maximizing mechanism for the seller in the first period, given that any buyer who does not obtain the first object will participate in the auction for the second object.

For any buyer who did not obtain the first object, it is a weakly dominant strategy in the second auction to submit a bid equal to his valuation. Further, the reserve price r in the second auction is fixed before the first item is sold. As a result, buyers have no incentive to bid untruthfully for the first item in order to affect behavior in the auction.

Without loss of generality, we restrict attention to direct mechanisms in which buyers report their types. Let $\mathbf{x} \in [\underline{x}, \bar{x}]^N$ denote the vector of *reported* types. A direct mechanism specifies, for any given $\mathbf{x}$, the probability that each bidder i gets the good $P_i(\mathbf{x}) \geq \mathbf{0}$ with $\Sigma_{i=1}^N P_i(\mathbf{x}) \leq \mathbf{1}$ and the payment $t_i(\mathbf{x})$ that he must make.

We will work quite a bit with order statistics. Order the valuations from highest to lowest $X_{(1)}, X_{(2)}, \ldots, X_{(N)}$. It will also be useful to define the order statistics of the competing valuations that a single buyer faces. Order the valuations of the other $N-1$ buyers from highest to lowest $Y_{(1)}, Y_{(2)}, \ldots, Y_{(N-1)}$.

3 The Optimal Mechanism

Fix a buyer i with valuation X_i. Buyer i's payoff in the second period, provided that he did not get the first object, depends on whether or not the first object was allocated to his competitor with the highest type $Y_{(1)}$. If so, then buyer i's payoff,

$$\max\{X_i - \max\left\{r, Y_{(2)}\right\}, 0\},$$

is a function of the highest remaining competitor's type $Y_{(2)}$. If not, then buyer i's payoff is

$$\max\left\{X_i - \max\left\{r, Y_{(1)}\right\}, 0\right\}.$$

Thus, the payoff to a buyer depends on the two highest valuations among his competitors, and on whether or not those valuations exceed r. We denote the highest-type competitor of bidder i by j (so that $X_j = Y_{(1)}$). Then the expected payoff to a bidder i with type $x_i \geq r$, given reports $\mathbf{x}$ and excluding any payment to the first seller, is

$$P_i(\mathbf{x}) \cdot x_i + P_j(\mathbf{x}) \cdot \max\left\{x_i - \max\left\{r, y_{(2)}\right\}, 0\right\}$$
$$+ (1 - P_i(\mathbf{x}) - P_j(\mathbf{x})) \cdot \max\left\{x_i - \max\left\{r, y_{(1)}\right\}, 0\right\}.$$

The payoff to a buyer i with type $x_i < r$ in the second period is 0, so the expected payoff given $\mathbf{x}$ is just $P_i(\mathbf{x}) \cdot x_i$.

It will be convenient notationally to re-specify payoffs and allocations in terms of the vector of reported realizations of order statistics. Given vector of reported types $\mathbf{x}$, let $\hat{\mathbf{x}}$ denote the corresponding vector of reported types ordered from highest to lowest (with ties broken arbitrarily), so that the k-th element of $\hat{\mathbf{x}}$, $\widehat{x}_k$, is the k-th highest reported type in $\mathbf{x}$, $x_{(k)}$. Let $\hat{\mathbf{f}}$ denote the joint density of $\widehat{\mathbf{x}}$. Similarly, define $\widehat{\mathbf{y}}$ as the ordered vector of competitors' reported types facing a single buyer, with joint density $\widehat{\mathbf{g}}$. Given a bidder's type x and competitors' types $\widehat{\mathbf{y}}$, let $(x; \widehat{\mathbf{y}})$ denote the ordered vector of all N types. For each $k \in \{1, \ldots, N\}$ and $\hat{\mathbf{x}}$, let $\hat{p}^k(\hat{\mathbf{x}})$ denote the probability that the mechanism assigns the object to the bidder with the k-th highest report.

Using this notation, we can write the interim payoff of a type-x buyer who reports truthfully when other buyers also report truthfully reports truthfully as follows, where for readability we omit the dependence of $\hat{p}^k$ on $(x; \widehat{\mathbf{y}})$:[2]

$$\Pi(x|x) = E_{\widehat{\mathbf{Y}}} \left[\begin{array}{ll} & \mathbf{1}_{r > \widehat{Y}_1} \cdot (\hat{p}^1 \cdot x + (1 - \hat{p}^1) \cdot [x - r]) \\ + & \mathbf{1}_{x > \widehat{Y}_1 \geq r > \widehat{Y}_2} \cdot \left(\hat{p}^1 \cdot x + \hat{p}^2 \cdot [x - r] + (1 - \hat{p}^1 - \hat{p}^2) \cdot \left[x - \widehat{Y}_1\right]\right) \\ + & \mathbf{1}_{x > \widehat{Y}_1 \geq \widehat{Y}_2 \geq r} \cdot \left(\hat{p}^1 \cdot x + \hat{p}^2 \cdot \left[x - \widehat{Y}_2\right] + (1 - \hat{p}^1 - \hat{p}^2) \cdot \left[x - \widehat{Y}_1\right]\right) \\ + & \mathbf{1}_{\widehat{Y}_1 \geq x, r > \widehat{Y}_2} \cdot (\hat{p}^1 \cdot [x - r] + \hat{p}^2 \cdot x) \\ + & \mathbf{1}_{\widehat{Y}_1 \geq x > \widehat{Y}_2 \geq r} \cdot \left(\hat{p}^1 \cdot \left[x - \widehat{Y}_2\right] + \hat{p}^2 \cdot x\right) \\ + & \sum_{k=2}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq x > \widehat{Y}_{k+1}} \cdot \left(\hat{p}^{k+1} \cdot x\right) \end{array} \right].$$

[2]For completeness, set $\widehat{y}_{k+1} = \underline{v}$ when $k = N - 1$.

The first three lines give the payoffs to the buyer when his type x is the highest, the fourth and fifth lines are the payoffs when x is the second highest type, and the last line is the payoff when x is the $(k+1)$-th highest type, $k \geq 2$.

A buyer of type $x < r$ is never going to win the second auction. Therefore, his expected payoff from reporting truthfully is

$$\Pi(x|x) = E_{\widehat{\mathbf{Y}}} \left[\begin{array}{cc} & \mathbf{1}_{x > \widehat{Y}_1} \cdot (\hat{p}^1 \cdot x) \\ + & \sum_{k=1}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq x > \widehat{Y}_{k+1}} \cdot \left(\hat{p}^{k+1} \cdot x\right) \end{array} \right].$$

More generally, in Appendix A.1 we derive the payoff $\Pi(q|x)$ to a buyer of type x who reports his type as q.

The next step is to use the first-order incentive compatibility constraints to express the transfer payments in terms of buyers' payoffs and the allocation rule, and then we find the allocation rule that maximizes the sum of payments. Let $t(q)$ denote the expected transfer to a seller from a buyer who reports q. From the envelope theorem, the equilibrium payoff to a buyer of type x is

$$U(x) = U(\underline{x}) + \int_{\underline{x}}^{x} \Pi_2(x'|x')dx', \tag{1}$$

where $\Pi_2(x|x)$ is the partial derivative of $\Pi(q|x)$ with respect to the second argument (the buyer's true type) evaluated at the truthful report. For $x \geq r$, it is given by

$$\Pi_2(x|x) = E_{\widehat{\mathbf{Y}}} \left[\mathbf{1}_{x > \widehat{Y}_1} + \mathbf{1}_{\widehat{Y}_1 \geq x > \widehat{Y}_2} \cdot \left(\hat{p}^1 + \hat{p}^2\right) + \sum_{k=2}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq x > \widehat{Y}_{k+1}} \cdot \hat{p}^{k+1} \right],$$

and for $x < r$ it is

$$\Pi_2(x|x) = E_{\widehat{\mathbf{Y}}} \left[\mathbf{1}_{x > \widehat{Y}_1} + \sum_{k=1}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq x > \widehat{Y}_{k+1}} \cdot \hat{p}^{k+1} \right].$$

That is, $\Pi_2(x|x)$ equals the equilibrium probability that a type-x buyer gets an object, either the first or the second. The *ex ante* expected buyer payoff is thus

$$E\left[U(X)\right] = \int_{\underline{x}}^{\bar{x}} \int_{\underline{x}}^{x} \Pi_2(x'|x')dx' f(x) = \int_{\underline{x}}^{\bar{x}} \tfrac{1-F(x)}{f(x)} \Pi_2(x|x) = E\left[\tfrac{1-F(X)}{f(X)} \Pi_2(X|X)\right],$$

so the expected transfer is

$$Et(X) = E\left[\Pi(X|X) - U(X)\right] = t(\underline{x}) - \Pi(\underline{x}|\underline{x}) + E\left[\Pi(X|X) - \tfrac{1-F(X)}{f(X)}\Pi_2(X|X)\right].$$

Plugging in the expressions for $\Pi(x|x)$ and $\Pi_2(x|x)$, we get

$$\begin{aligned} Et(X) &= t(\underline{x}) - \Pi(\underline{x}|\underline{x}) \\ &+ E\left[\begin{array}{l} \mathbf{1}_{\widehat{X}_1 = X} \cdot \left(\begin{array}{c} \psi(\widehat{X}_1) - \max\left\{r, \widehat{X}_2\right\} + \hat{p}^1 \cdot \left[\max\left\{r, \widehat{X}_2\right\}\right] \\ +\hat{p}^2 \cdot \left[\max\left\{r, \widehat{X}_2\right\} - \max\left\{r, \widehat{X}_3\right\}\right] \end{array}\right) \\ + \ \mathbf{1}_{\widehat{X}_2 = X} \cdot \left(\hat{p}^1 \cdot \left[\psi(\widehat{X}_2) - \max\left\{r, \widehat{X}_3\right\}\right] + \hat{p}^2 \cdot \psi(\widehat{X}_2)\right) \\ + \ \sum_{k=3}^{N} \mathbf{1}_{\widehat{X}_k = X} \cdot \left(\hat{p}^k \cdot \psi(\widehat{X}_3)\right) \end{array}\right]. \end{aligned}$$

Because the probability that a given bidder has the k-th highest value is $1/N$ for each $k \in \{1, \ldots, N\}$, we can rewrite the expected transfer as

$$\begin{aligned} Et(X) &= t(\underline{x}) - \Pi(\underline{x}|\underline{x}) \\ &+ \tfrac{1}{N} E \left(\begin{array}{c} \psi(\widehat{X}_1) - \max\left\{r, \widehat{X}_2\right\} + \hat{p}^1 \cdot \left[\max\left\{r, \widehat{X}_2\right\}\right] \\ +\hat{p}^2 \cdot \left[\max\left\{r, \widehat{X}_2\right\} - \max\left\{r, \widehat{X}_3\right\}\right] \end{array}\right) \\ &+ \tfrac{1}{N} E \left(\hat{p}^1 \cdot \left[\psi(\widehat{X}_2) - \max\left\{r, \widehat{X}_3\right\}\right] + \hat{p}^2 \cdot \psi(\widehat{X}_2)\right) \\ &+ \sum_{k=3}^{N} \tfrac{1}{N} E\left(\hat{p}^k \cdot \psi(\widehat{X}_k)\right). \end{aligned} \tag{2}$$

The seller maximizes expected revenue $ER(\widehat{X}) = N \cdot Et(\widehat{X})$ subject to incentive compatibility and individual rationality. Maximizing that integral pointwise yields a solution that may fail to be globally incentive compatible (a buyer may prefer to report a type far from his own), as we will see.

Given any vector of ordered types $\hat{\mathbf{x}}$, taking the derivative of the seller's expected revenue with respect to $\hat{p}^k(\hat{\mathbf{x}})$ yields the following:

1. If $\widehat{x}_2 \geq r$,

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^1(\hat{\mathbf{x}})} = \frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})} = [\psi(\widehat{x}_2) + \widehat{x}_2 - \max\{\widehat{x}_3, r\}]\, \widehat{\mathbf{f}}(\widehat{\mathbf{x}}),$$

   and for all $k > 2$,

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} = \psi(\widehat{x}_k)\, \widehat{\mathbf{f}}(\widehat{\mathbf{x}}).$$

2. If $\widehat{x}_1 \geq r > \widehat{x}_2$,

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^1(\hat{\mathbf{x}})} = r\widehat{\mathbf{f}}(\widehat{\mathbf{x}})$$

   and for all $k > 1$,

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} = \psi(\widehat{x}_k)\, \widehat{\mathbf{f}}(\widehat{\mathbf{x}}).$$

3. If $r > \widehat{x}_1$, for all k,

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} = \psi(\widehat{x}_k)\, \widehat{\mathbf{f}}(\widehat{\mathbf{x}}).$$

Global incentive compatibility is satisfied if a bidder cannot increase his probability of getting an item (either the first or the second) by underreporting his type, or decrease the probability by overreporting his type. Formally, the condition is that for any type x and any reports q, q' such that $q > x > q'$, we have $\Pi_2(q|x) \geq \Pi_2(x|x) \geq \Pi_2(q'|x)$. We find that the solution to this pointwise maximization satisfies that condition when $r \in [\psi^{-1}(0), \bar{x})$ but not when $r \in (\underline{x}, \psi^{-1}(0))$.

The source of the problem is a discontinuity in the marginal revenue expressions above. If $\widehat{x}_2 \geq r$, then the marginal revenue from allocating to the highest or second-highest bidder is $\psi(\widehat{x}_2) + \widehat{x}_2 - \max\{\widehat{x}_3, r\}$. If $\widehat{x}_1 \geq r > \widehat{x}_2$, however, then the marginal revenue from allocating to the highest bidder is r, independent of the exact values of $\widehat{x}_2$ and $\widehat{x}_3$. If $\psi(r) < 0$, then as $\widehat{x}_2$ moves from just below r to just above r, marginal revenue jumps from strictly positive to strictly negative. That downward switch drives the failure of global incentive compatibility, which we explore below.

3.1 If $r \in \left[\psi^{-1}(0), \bar{x}\right)$

In the monopoly case, the optimal mechanism is to allocate the object to the bidder with the highest valuation if and only if $\psi(\widehat{x}_1) \geq 0$. Thus, if $r \in [\psi^{-1}(0), \bar{x})$, then the second seller is using a reserve price higher than the optimal reserve price in the standard mechanism design setting. In this case, the solution to the first seller's pointwise maximization problem is to

- allocate to the top bidder if $\widehat{x}_1 \geq r > \widehat{x}_2$ or $r > \widehat{x}_1 \geq \psi^{-1}(0)$, because the marginal revenue for $\hat{p}^1$ is $r > 0$ or $\psi\left(\widehat{x}_1\right) > 0$, respectively;

- allocate to one of the top two bidders if $\widehat{x}_2 \geq r$, because the marginal revenue for both $\hat{p}^1$ and $\hat{p}^2$ is $\psi\left(\widehat{x}_2\right) + \widehat{x}_2 - \max\left\{\widehat{x}_3, r\right\} \geq \psi\left(\widehat{x}_2\right) > 0$;

and not to allocate otherwise. If the seller uses this rule (and any method of breaking indifferences between allocating to the highest and second-highest bidders), then it is straightforward to show that the probability of getting an item (first or second) is increasing in the report. Thus, global incentive compatibility is satisfied.

Theorem 1. *If the reserve price in the second auction is $r \in [\psi^{-1}(0), \bar{x})$, then the following is an optimal (direct) mechanism for the first seller. (Ties are broken randomly.)*

Allocation rule: *The seller allocates the good if and only if $\psi\left(x_{(1)}\right) \geq 0$; allocation in that case is to the bidder with the highest valuation if $x_{(2)} < r$, and it is to either the bidder with the highest valuation or the bidder with the second-highest valuation if $x_{(2)} \geq r$.*

Transfers:

1. *If $\psi\left(x_{(1)}\right) \geq 0$ and $x_{(2)} < r$, then the bidder with the highest valuation pays $\max\left\{\psi^{-1}(0), x_{(2)}\right\}$ and the other bidders pay nothing.*

2. *If $x_{(2)} \geq r$, then the bidder who receives the object pays $\max\left\{r, x_{(3)}\right\}$ and the other bidders pay nothing.*

3. *If the good is not allocated, then there are no payments.*

The mechanism in Theorem 1 can be implemented through a hybrid second- and third-price auction with reserve prices: if the highest bid is above $\psi^{-1}(0)$, then the item goes to the highest bidder at a price equal to either 1) the maximum of $\psi^{-1}(0)$ and the second-highest bid if that second-highest bid is below r, or 2) the maximum of r and the third-highest bid if the second-highest bid is above r.

3.2 If $r \in \left(\underline{x}, \psi^{-1}(0)\right)$

In this case, the second seller uses a reserve price lower than the optimal reserve price in the standard mechanism design setting, and the solution to the pointwise maximization problem turns out to violate incentive compatibility. The following notation will be useful.

Definition 1. For $x \in [\underline{x}, \bar{x}]$, define $a(x) \in [\underline{x}, \bar{x}]$ as

$$a(x) \equiv \min \{a \geq x : a + \psi(a) \geq x\}.$$

Note that when $\psi(r) < 0$, $a(r)$ is the valuation that solves $a + \psi(a) = r$. In that case, $a(r) > r$ and $a(r) < \psi^{-1}(0)$. Given this definition, the solution to the pointwise maximization problem is to

- allocate to the top bidder if $\widehat{x}_1 \geq r > \widehat{x}_2$;
- allocate to one of the top two bidders if $\widehat{x}_3 \geq r$ and $\widehat{x}_2 + \psi(\widehat{x}_2) - \widehat{x}_3 \geq 0$;
- allocate to one of the top two bidders if $\widehat{x}_2 \geq a(r)$ and $r > \widehat{x}_3$, because the marginal revenue for both $\hat{p}^1$ and $\hat{p}^2$ is $\widehat{x}_2 + \psi(\widehat{x}_2) - r \geq a(r) + \psi(a(r)) - r = 0$;
- not allocate if $a(r) > \widehat{x}_2$ and $r > \widehat{x}_3$, because the marginal revenue for both $\hat{p}^1$ and $\hat{p}^2$ is $\widehat{x}_2 + \psi(\widehat{x}_2) - r < a(r) + \psi(a(r)) - r = 0$;

and not to allocate otherwise.

This rule is not incentive compatible. To see why, suppose that bidder i with type x between r and $a(r)$ considers deviating to a report q below r. If x is the highest type, then bidder i is certain to get an item with either report: the first item if he reports x, the second item if he reports q. If x is the third-highest or lower order type, then bidder i will get nothing with either report. He will also get nothing if x is the second-highest type and $\widehat{x}_3$ is greater than r, because the first unit will not be allocated at either report (the marginal revenue from doing so is negative), so bidder i loses the second auction to the bidder with the highest type. The non-monotonicity arises when x is the second-highest type and $\widehat{x}_3$ is less than r. If bidder i reports truthfully, then the first unit is not allocated and he loses the second auction to the bidder with the highest type. But if he reports a type below r, then the rule above specifies that the highest bidder gets the first unit, and then bidder i will win the second. Thus, a bidder with a type between r and $a(r)$ is more likely to get an item by reporting a type below r than by reporting truthfully.

More formally, incentive compatibility requires the second-order condition that

$$\int_q^x \Pi_2(x'|x')dx' \geq \int_q^x \Pi_2(q|x')dx', \tag{3}$$

for all $x, q \in [\underline{x}, \bar{x}]$, where $\Pi_2(q|x)$, the derivative of the gross payoff $\Pi(q|x)$ with respect to the buyer's true type, corresponds to the probability that buyer of type x gets an item (either the first or the second) when reporting type q. (See Section A.3.) The allocation rule derived above violates that condition at $x = r$ and any $q < r$.

We proceed to find the optimal mechanism through a process of "guess and verify." First, we guess that the constraints in Expression 3 bind only for types x below $a(r)$; that for type r the constraint binds only for underreports $q < r$; and that for the rest of the types $x \leq a(r)$ the constraint binds only for a marginal underreport, $q = x - \epsilon$ for vanishingly small ϵ. Those guesses yield a continuum of constraints that take the form

$$\int_q^r \Pi_2(x'|x')dx' \geq \int_q^r \Pi_2(q|x')dx' \tag{4}$$

for each $q \in [\underline{x}, r)$ (let $\lambda_{r,q}$ denote the corresponding Lagrange multiplier); and

$$\Pi_2(x|x) \geq \limsup_{\epsilon \searrow 0} \Pi_2(x - \epsilon|x) \tag{5}$$

for each $x \in (\underline{x}, a(r)]$ (multiplier μ_x). The constraints reflect the idea that a buyer must have a weakly higher chance of getting an object if he reports truthfully than if he underreports.[3]

We derive first-order conditions by maximizing Expression 2 subject to the constraints in Expressions 4 and 5. We then guess the values of $\lambda_{r,q}$ and μ_x, derive the solution using those guesses, and show that it satisfies incentive compatibility. (See Appendix A.) The optimal mechanism corresponds to pointwise maximization except when there are either one or two bids above the reserve price r. When $\widehat{x}_1 > r > \widehat{x}_3$ and $\widehat{x}_2$ is either below r or just above it (between r and $a(r)$), then the constraints in Expressions 4 and 5 bind and the allocation rule needs to be adjusted. Depending on the value of r, the solution is to allocate either in all of these cases (regardless of the exact valuations of the bidders) or in none of them.

The intuition for this solution is as follows. We would like to allocate the good to the highest bidder when $\widehat{x}_1 > r > \widehat{x}_2$, because the marginal revenue r from doing so is positive, but not when $a(r) > \widehat{x}_2 \geq r > \widehat{x}_3$, because in this case the marginal revenue is negative (i.e., $\widehat{x}_2 + \psi(\widehat{x}_2) - r < 0$). Roughly, the constraints in Expression 4 mean that if we allocate when $\widehat{x}_1 = x^*$ and $\widehat{x}_2 < r$, then we also have to allocate when $\widehat{x}_1 = x^*$ and $\widehat{x}_2 = r$: otherwise a bidder with type r would be more likely to get an item by underreporting. The constraints in Expression 5 then imply that we must also allocate when $\widehat{x}_2$ is just above r, and then when $\widehat{x}_2$ is just above that value, and so on. Iterating those constraints, we conclude that if we allocate when $\widehat{x}_1 = x^*$ and $\widehat{x}_2 < r$, then we must also allocate when we replace $\widehat{x}_2$ with any higher value, including values above x^*: that is, allocate whenever $\widehat{x}_1 \geq x^*$.

The seller's maximization problem, then, consists of finding the optimal cutoff x^* such that when $\widehat{x}_3 < r$ and $\widehat{x}_2 < a(r)$, the seller allocates if and only if $\widehat{x}_1 \geq x^*$. The resulting revenue is $N \cdot F(r)^{N-2}$ times $Z^r(x^*)$, where the function $Z^r(x^*)$ is defined as follows:

[3]Note that for type r, there is a redundancy: μ_r and $\lambda_{r,r-0}$ refer to the same constraint. In the proofs, it will be convenient notationally to use one constraint in some cases and the other in other cases.

Definition 2. For $r \in [\underline{x}, \psi^{-1}(0))$ and $x^* \in [r, \bar{x}]$, define

$$\begin{aligned} Z^r(x^*) &\equiv \tfrac{1}{N \cdot F(r)^{N-2}} E\left[\mathbf{1}_{\widehat{X}_1 \geq r > \widehat{X}_2} \cdot r + \mathbf{1}_{\widehat{X}_1 \geq r, a(r) > \widehat{X}_2 \geq r > \widehat{X}_3} \cdot \left(\psi\left(\widehat{X}_2\right) + \widehat{X}_2 - r\right)\right] \\ &= rF(r)\left[1 - F(x^*)\right] + (N-1) \int_{x^*}^{\bar{x}} \left(\int_{r}^{\min\{x, a(r)\}} [\psi(x') + x' - r] f(x') dx' \right) f(x) dx. \end{aligned}$$

To interpret $Z^r(x^*)$, note that the revenue $N \cdot F(r)^{N-2} Z^r(x^*)$ is the marginal revenue r when $\widehat{x}_1 \geq x^*$ and $\widehat{x}_2 < r$ times the probability of that event, plus the (negative) expected marginal revenue when $\widehat{x}_1 \geq x^*$ and $a(r) > \widehat{x}_2 \geq r > \widehat{x}_3$ times the probability of *that* event. (Recall that when $\widehat{x}_2 \geq a(r)$, we want to allocate regardless, so we do not need to include that case in the revenue maximization.) The function $Z^r(x^*)$ is quasiconvex, so the optimal x^* is at a corner: either $x^* = r$ or $x^* = \bar{x}$. Observe that $Z^r(\bar{x}) = 0$ (because $\widehat{x}_1$ cannot exceed $\bar{x}$). Thus, $x^* = \bar{x}$ is optimal if and only if $Z^r(r) \leq 0$, because then revenue is higher at $x^* = \bar{x}$ than at $x^* = r$. If $Z(r) \geq 0$, then $x^* = r$ is optimal. That logic forms the basis for our guesses of the values of the Lagrange multipliers.

The value of $Z^r(r)$ is decreasing in the number of bidders N. When N is large enough, all else equal, the optimal cutoff x^* equals $\bar{x}$, and the item is not allocated unless the second-highest bid is at least $a(r)$. We have not been able to establish whether or not $Z^r(r)$ is monotonic in the reserve price r. We do know, however, that for small enough values of r, again the optimal cutoff x^* equals $\bar{x}$. The reason is that $Z^r(r)$ is continuous and strictly negative at $r = \underline{x}$. For values of r such that $Z^r(r) < 0$, the seller allocates if and only if

$$\psi\left(\widehat{x}_2\right) + \widehat{x}_2 - \max\left\{r, \widehat{x}_3\right\} \geq 0. \tag{6}$$

When $Z^r(r) > 0$, then the seller allocates either if Condition 6 holds or if $\widehat{x}_1 \geq r > \widehat{x}_3$.

We summarize the optimal mechanism in the following two theorems.

Theorem 2. *If the reserve price in the second auction is* $r \in (\underline{x}, \psi^{-1}(0))$*, and* $Z^r(r) \leq 0$*, then the following is an optimal (direct) mechanism for the first seller. (Ties are broken randomly.)*

Allocation rule: *The seller allocates the good to the bidder with the second-highest valuation if* $\psi(\widehat{x}_2) + \widehat{x}_2 - \max\{r, \widehat{x}_3\} \geq 0$, *and does not allocate otherwise.*

Transfers:

1. *If* $x_{(3)} \leq r$ *and the good is allocated* ($x_{(2)} \geq a(r)$), *then the bidder with the highest valuation pays* $a(r) - r > 0$, *the bidder with the second-highest valuation pays* $a(r) > 0$, *and the other bidders pay nothing.*

2. *If* $x_{(3)} \in (r, \psi^{-1}(0))$ *and the good is allocated* ($x_{(2)} \geq a(x_{(3)})$), *then the bidder with the highest valuation pays* $a(x_{(3)}) - x_{(3)} > 0$, *the bidder with the second-highest valuation pays* $a(x_{(3)}) > 0$, *and the other bidders pay nothing.*

3. *If* $\psi(x_{(3)}) \geq 0$ *(in which case the good is allocated because* $x_{(2)} \geq a(x_{(3)}) = x_{(3)}$*), then the bidder with the second-highest valuation pays* $x_{(3)} > 0$ *and the other bidders pay nothing.*

4. *If the good is not allocated, then there are no payments.*

Theorem 3. *If the reserve price in the second auction is* $r \in (\underline{x}, \psi^{-1}(0))$, *and* $Z^r(r) \geq 0$, *then the following is an optimal (direct) mechanism for the first seller. (Ties are broken randomly.)*

Allocation rule: *The seller allocates to the bidder with the highest valuation if* $\widehat{x}_1 \geq r > \widehat{x}_2$, *allocates to either the bidder with the highest valuation or the bidder with the second-highest valuation if* $\widehat{x}_2 \geq r > \widehat{x}_3$, *allocates to the bidder with the second-highest valuation if* $\widehat{x}_3 \geq r$ and $\psi(\widehat{x}_2) + \widehat{x}_2 - \widehat{x}_3 \geq 0$, *and does not allocate otherwise.*

Transfers:

1. *If* $\widehat{x}_1 \geq r > \widehat{x}_2$, *then the bidder with the highest valuation pays* r *and the other bidders pay nothing.*

2. *If* $\widehat{x}_2 \geq r > \widehat{x}_3$, *then the bidder who receives the object pays* r *and the other bidders pay nothing.*

3. If $x_{(3)} \in (r, \psi^{-1}(0))$ *and the good is allocated* ($x_{(2)} \geq a\left(x_{(3)}\right)$)*, then the bidder with the highest valuation pays* $a\left(x_{(3)}\right) - x_{(3)} > 0$*, the bidder with the second-highest valuation pays* $a\left(x_{(3)}\right) > 0$*, and the other bidders pay nothing.*

4. If $\psi\left(x_{(3)}\right) \geq 0$ *(in which case the good is allocated because* $x_{(2)} \geq a\left(x_{(3)}\right) = x_{(3)}$*), then the bidder with the second-highest valuation pays* $x_{(3)} > 0$ *and the other bidders pay nothing.*

5. If the good is not allocated, then there are no payments.

This mechanism is similar to the optimal mechanism derived in Hendricks and Wiseman [forthcoming] for the special case of no reserve price in the second auction, but it has some interesting new features. The optimal withholding rule now is a function of the first-, second-, and third-highest values, rather than just the second- and third-highest. Further, the withholding rule now varies with the number of bidders, unlike both the Hendricks and Wiseman [forthcoming] case and the standard auction environment: the value of $Z^r(r)$ is decreasing in N.

4 Implementing the Optimal Mechanism

In this section, we show that the optimal mechanism can be implemented with a modified third-price auction. For simplicity, we focus on the case where $Z^r(r) \leq 0$ (for example, where there is a low reserve price $r \approx \underline{x}$ in the second auction), but the arguments extend to the case $Z^r(r) > 0$.

In order to implement the payments and allocation rule from Theorem 2 define the *modified third-price auction* as follows: each buyer submits a bid in $[\underline{x}, \bar{x}]$. As a function of the vector of bids $\mathbf{b}$, the good is allocated to the second-highest bidder if and only if

$$\psi\left(b_{(2)}\right) + b_{(2)} \geq \max\left\{r, b_{(3)}\right\}).$$

If the unit is not allocated, then no one makes any payments. If the unit is allocated, then the payments are based on the reserve price and the third-highest bid, $b_{(3)}$.

When $\psi(b_{(3)}) > 0$, the highest bidder pays nothing and the second-highest bidder pays $b_{(3)}$; when $\psi(b_{(3)}) < 0$, then the highest bidder pays

$$a\left(\max\left\{r, b_{(3)}\right\}\right) - \max\left\{r, b_{(3)}\right\} > 0$$

and the second-highest bidder pays $a\left(\max\left\{r, b_{(3)}\right\}\right)$.

Theorem 4. *If the reserve price in the second auction is $r \in (\underline{x}, \psi^{-1}(0))$, and $Z^r(r) \leq 0$, then truthful bidding is an equilibrium of the modified third-price auction, and that equilibrium yields the optimal expected revenue for the first seller.*

Consider, for example, the highest-valuation buyer in the case where $x_{(3)} < r$ and the item is allocated $(x_{(2)} \geq a(r))$. Truthfully bidding $b = x_{(1)}$ yields a payoff of

$$x_{(1)} - r - [a(r) - r] = x_{(1)} - a(r);$$

the bidder transfers $a(r) - r)$ to the first seller and then wins the second auction at the reserve price r. Any bid above $x_{(2)}$ yields that same payoff. A bid between $a(r)$ and $x_{(2)}$ also results in payoff $x_{(1)} - a(r)$: the bidder gets the first item and transfers $a(r)$ to the first seller. Any bid below $a(r)$ gives a lower payoff, $x_{(1)} - x_{(2)}$, because the first item will not be allocated, no transfers will be made to the first seller, and the bidder will win the second item at price $x_{(2)}$. The other cases are similar.

5 Equilibrium

In this section, we consider a simple form of strategic interaction between sellers, where the second seller chooses a reserve price and the first seller best responds as in Theorems 1, 2, and 3. We study whether or not an equilibrium exists and what we can say about its properties. In particular, we ask whether the equilibrium reserve price r^* is less than $\psi^{-1}(0)$. If so, then the first seller's equilibrium mechanism will be impossible to implement with a standard auction.

We begin by computing the equilibrium in an example where there are three buyers whose valuations are distributed uniformly between zero and one. That is,

$N = 3$ and $F(x) = x$]. In that case, virtual valuations are given by $\psi(x) = 2x - 1$, so $\psi^{-1}(0) = 1/2$. Furthermore, for $r \in (0, 1/2)$, we have that

$$a(r) = (r + 1)/3$$

and

$$Z^r(r) = -\frac{1}{27}(1 - 2r)^2 (8 - 7r) + r^2(1 - r).$$

That value is negative (and thus the optimal allocation rule for the first seller is $x^* = 1$) when $r \leq \tilde{r} \approx 0.263$, where $\tilde{r}$ is the solution to $r^3 - 33r^2 + 39r - 8 = 0$. When $\frac{1}{2} > r \geq \tilde{r}$, the optimal allocation rule for the first seller is $x^* = r$. For $r \geq \frac{1}{2}$, the first seller allocates whenever the highest valuation is above $\frac{1}{2}$.

We can then calculate the expected revenues of the second seller (see appendix for details) as a function of her reserve price r, given that the first seller's mechanism is a best reply. Her expected revenue is

$$R_2(r) = \frac{1}{4} + \frac{3}{2}r^2 - 4r^3 + \frac{9}{4}r^4$$

when $r \in [1/2, 1]$; it is

$$R_2(r) = \frac{125}{432} - \frac{7}{27}r + \frac{19}{9}r^2 - \frac{124}{27}r^3 + \frac{263}{108}r^4$$

when $r \in [\tilde{r}, 1/2)$; and it is

$$R_2(r) = \frac{125}{432} + \frac{8}{9}r^2 + \frac{5}{27}r^3 - \frac{47}{36}r^4$$

when $r \in [0, \tilde{r}]$. The graph of the revenue function is illustrated in Figure 1.

The first point to note is that the revenue function is continuous at $r = 1/2$. The reason is that the allocation probabilities for the first and second units are continuous in r. The second point to note is that the revenue function exhibits a downward discontinuity at $\tilde{r}$. This discontinuity comes from the discrete increase in the probability of the first good being allocated when x^* jumps from $x^* = 1$ to $x^* = \tilde{r}$. All else equal,

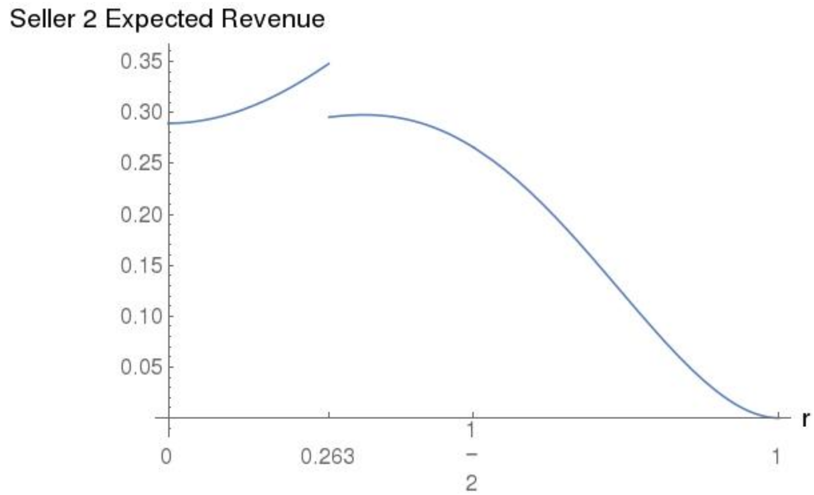

Figure 1: Seller 2's revenue as a function of r

the second seller does better when the first seller does not allocate because then she has a higher chance of earning $\max\{r, x_{(2)}\}$ instead of $\max\{r, x_{(3)}\}$.

Revenue in our example has two local maxima. Above $\tilde{r}$, revenue of 0.298 is attained at $r \approx 0.320$. Revenue is increasing in r on $[0, \tilde{r}]$, so the maximum value, 0.341, is attained at $\tilde{r}$. Thus, in equilibrium the second seller chooses reserve price $r^* = \tilde{r} \approx 0.263$ and the first seller responds as in Theorem 2. That is, the first seller allocates if $3x_{(2)} - 1 \geq \max\{r^*, x_{(3)}\}$ and not otherwise. The resulting expected revenue for the first seller is approximately 0.343.

When there is no reserve price in the second auction, Hendricks and Wiseman ([forthcoming]) calculate that the expected revenue of the first seller in this example is approximately 0.382 and the expected revenue of the second seller is approximately 0.289. Thus, relative to using no reserve price, the second seller increases her revenue by 18% by setting r^*, while reducing the first seller's revenue by 10%. The fact that raising the reserve price in the second auction, which *reduces* the competition facing the first seller, turns out to lower the first seller's revenue may seem counterintuitive. The explanation is that the reserve price also reduces the expected surplus available to buyers in the second auction. Because the first seller appropriates some of that surplus through the threat of withholding the first object, her revenue falls.

In equilibrium, the negative effect of surplus reduction dominates the positive effect of reduced competition in this example.

In a monopoly setting, the seller sets the optimal reserve price to equate the expected loss from not selling to the expected gain from selling at a higher price. This tradeoff is determined solely by the distribution of buyer values F. However, in our model, the distribution of buyer values that the second seller faces is endogenous, and she has to take into account how her reserve price affects the probability that the first seller will allocate her unit. In the uniform example above, we quantify those tradeoffs and calculate the optimal reserve price for the second seller. It is not clear what we can say more generally. The second seller will certainly want to avoid the jump in the allocation probability when the first seller's optimal cutoff switches from $x^* = \bar{x}$ to $x^* = r$, but exactly where the equilibrium reserve price lies is likely to depend on the properties of F. We can show, however, that an equilibrium exists and that the first seller uses a withholding rule that cannot be implemented with a simple reserve price (as long as the optimal static reserve price is nontrivial).[4]

Theorem 5. *An equilibrium of the game between sellers exists. If in addition* $\psi^{-1}(0) > \underline{x}$*, then seller 2's equilibrium reserve price* $r^* < \psi^{-1}(0)$.

For intuition, recall that $\psi^{-1}(0)$ is the optimal reserve price for a single seller facing distribution F. The second seller in our setting chooses r below that level because she faces a worse distribution of buyer values: the highest or second-highest bidder may have already obtained an item from the first seller. At $r = \psi^{-1}(0)$, lowering r has a first order positive effect on the second seller's revenue conditional on the first seller's allocation rule and only a second order effect on that allocation rule.

6 Concluding Remarks

We analyze the outcome of competition in mechanisms when sales are sequential, in a setting where 1) there are two sellers, 2) the second seller choose the reserve

[4]If the minimum valuation $\underline{x}$ has a positive virtual valuation, then it is an equilibrium for both sellers to allocate to the highest bidder.

price in a second-price auction while the first seller may use any mechanism, and 3) the second seller is the first mover, committing to a reserve price that the first seller then best responds to. We hope to generalize our model in all three directions. Information leakage, which is conceptually distinct from the spillover effect between sequential sellers that drives our analysis, becomes a complication. In general, the best response for the second seller depends on what information about the bidders' types is disclosed after the first period, and buyers may worry that information about their types revealed in their bids will influence the future bids of their competitors.

Allowing the second seller to choose any ex post incentive compatible mechanism would relax the restriction on her strategy space while preserving robustness to the possibility of information leakage, so that we could consider competition under other ways of modeling the timing of moves and information revelation. Similarly, in the case of more than two sellers, restricting later sellers to ex post incentive compatible mechanisms may be a tractable way to separate the effects of sequential competition in mechanisms from those of information leakage.

A Proving Theorems 1, 2, and 3

A.1 Payoff from false report

We derive the payoff to a type-x buyer who reports his type as q. If $x \geq r$ and $q \geq x$, then

$$
\begin{aligned}
\Pi(q|x) &= \int_{[\underline{x},\bar{x}]^{N-1}:x>\widehat{y}_1} \begin{pmatrix} x - \max\{\widehat{y}_1, r\} + \hat{p}^1((q;\widehat{\mathbf{y}})) \cdot \max\{\widehat{y}_1, r\} \\ +\hat{p}^2((q;\widehat{\mathbf{y}})) \cdot [\max\{\widehat{y}_1, r\} - \max\{\widehat{y}_2, r\}] \end{pmatrix} \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\
&+ \int_{[\underline{x},\bar{x}]^{N-1}:q>\widehat{y}_1\geq x>\widehat{y}_2} \left(\hat{p}^1((q;\widehat{\mathbf{y}})) \cdot x + \hat{p}^2((q;\widehat{\mathbf{y}})) \cdot [x - \max\{\widehat{y}_2, r\}]\right) \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\
&+ \int_{[\underline{x},\bar{x}]^{N-1}:q>\widehat{y}_1>\widehat{y}_2\geq x} \left(\hat{p}^1((q;\widehat{\mathbf{y}})) \cdot x\right) \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\
&+ \int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1\geq q\geq x>\widehat{y}_2} \left(\hat{p}^1((q;\widehat{\mathbf{y}})) \cdot [x - \max\{\widehat{y}_2, r\}] + \hat{p}^2((q;\widehat{\mathbf{y}})) \cdot x\right) \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\
&+ \sum_{k=1}^{N-1} \left[\int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1},\widehat{y}_2\geq x} \left(\hat{p}^{k+1}((q;\widehat{\mathbf{y}})) \cdot x\right) \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \right].
\end{aligned}
$$

If $x \geq r$ and $q < x$, then

$$
\begin{aligned}
\Pi(q|x) &= \int_{[\underline{x},\bar{x}]^{N-1}:q>\widehat{y}_1} \begin{pmatrix} x - \max\{\widehat{y}_1, r\} + \hat{p}^1((q;\widehat{\mathbf{y}})) \cdot \max\{\widehat{y}_1, r\} \\ +\hat{p}^2((q;\widehat{\mathbf{y}})) \cdot [\max\{\widehat{y}_1, r\} - \max\{\widehat{y}_2, r\}] \end{pmatrix} \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\
&+ \sum_{k=1}^{N-1} \left[\int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1},x>\widehat{y}_1} \begin{pmatrix} x - \max\{\widehat{y}_1, r\} \\ +\hat{p}^1((q;\widehat{\mathbf{y}})) \cdot [\max\{\widehat{y}_1, r\} - \max\{\widehat{y}_2, r\}] \\ +\hat{p}^{k+1}((q;\widehat{\mathbf{y}})) \cdot \max\{\widehat{y}_1, r\} \end{pmatrix} \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \right] \\
&+ \sum_{k=1}^{N-1} \left[\int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1},\widehat{y}_1\geq x>\widehat{y}_2} \begin{pmatrix} \hat{p}^1((q;\widehat{\mathbf{y}})) \cdot [x - \max\{\widehat{y}_2, r\}] \\ +\hat{p}^{k+1}((q;\widehat{\mathbf{y}})) \cdot x \end{pmatrix} \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \right] \\
&+ \sum_{k=2}^{N-1} \left[\int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1},\widehat{y}_2\geq x} \left(\hat{p}^{k+1}((q;\widehat{\mathbf{y}})) \cdot x\right) \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \right].
\end{aligned}
$$

If $x < r$, then

$$\Pi(q|x) = \int\limits_{[\underline{x},\bar{x}]^{N-1}:q>\widehat{y}_1} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right)\cdot x\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) + \sum_{k=1}^{N-1}\left[\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1}} \left(\hat{p}^{k+1}\left((q;\widehat{\mathbf{y}})\right)\cdot x\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right].$$

A.1.1 $\Pi_2(q|x)$

The derivative of the payoff with respect to its second argument (the buyer's true type), $\Pi_2(q|x)$, will be used below. If $x \geq r$ and $q \geq x$, then we calculate that derivative as

$$\begin{aligned} \Pi_2(q|x) &= \int\limits_{[\underline{x},\bar{x}]^{N-1}:x>\widehat{y}_1} \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) + \int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1\geq x>\widehat{y}_2} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right) + \hat{p}^2\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ &+ \int\limits_{[\underline{x},\bar{x}]^{N-1}:q>\widehat{y}_1>\widehat{y}_2\geq x} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ &+ \sum_{k=1}^{N-1}\left[\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1},\widehat{y}_2\geq x} \left(\hat{p}^{k+1}\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right]. \end{aligned}$$

If $x \geq r$ and $q < x$, then

$$\begin{aligned} \Pi_2(q|x) &= \int\limits_{[\underline{x},\bar{x}]^{N-1}:x>\widehat{y}_1} \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) + \sum_{k=1}^{N-1}\left[\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1},\widehat{y}_1\geq x>\widehat{y}_2} \begin{pmatrix} \hat{p}^1\left((q;\widehat{\mathbf{y}})\right) \\ +\hat{p}^{k+1}\left((q;\widehat{\mathbf{y}})\right) \end{pmatrix}\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \\ &+ \sum_{k=2}^{N-1}\left[\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1},\widehat{y}_2\geq x} \left(\hat{p}^{k+1}\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right]. \end{aligned}$$

If $x < r$, then

$$\Pi_2(q|x) = \int\limits_{[\underline{x},\bar{x}]^{N-1}:q>\widehat{y}_1} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) + \sum_{k=1}^{N-1}\left[\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1}} \left(\hat{p}^{k+1}\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right].$$

A.2 Convexity

We show that the payoff $\Pi(q|x)$ is convex in its second argument (the buyer's true type). It follows that $U(x)$, as the maximum of convex functions, is also convex. It is therefore absolutely continuous and so differentiable almost everywhere, and thus Expression 1 is valid.

The derivative of $\Pi(q|x)$ with respect to the buyer's type x corresponds to the probability that the buyer gets an item (either the first or the second). Conditional on the report, that probability is increasing in x because a buyer with a higher valuation is more likely to win the second auction if he does not win the first item. Formally, the second derivative of the payoff $\Pi(q|x)$ with respect to the buyer's true type, $\Pi_{22}(q|x)$, when $x \geq r$ and $q \geq x$ is given by

$$\begin{aligned}\Pi_{22}(q|x) \;&=\; \int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1=x} \left(1-\hat{p}^1\left((q;\widehat{\mathbf{y}})\right)-\hat{p}^2\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ &+ \int\limits_{[\underline{x},\bar{x}]^{N-1}:q>\widehat{y}_1,\widehat{y}_2=x} \left(\hat{p}^2\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) + \int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1\geq q,\widehat{y}_2=x} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}).\end{aligned}$$

The first integral represents the increase in the chance of getting an item when the buyer's type moves from just below the highest competitor's type $\widehat{y}_1$ to just above it: if $x > \widehat{y}_1$, then the buyer gets an item for sure because he would win the second auction. If $x < \widehat{y}_1$, then he gets an item only if he or the highest competitor gets the first item. Similarly, the second and third integrals represent the increase in the chance of getting an item when the buyer's type moves from just below the second highest competitor's type $\widehat{y}_2$ to just above it. Each of the three integrals is weakly positive, so $\Pi_{22}(q|x) \geq 0$.

Analogously, when $x \geq r$ and $q < x$, $\Pi_{22}(q|x)$ is given by

$$\begin{aligned}\Pi_{22}(q|x) \;&=\; \sum_{k=1}^{N-1}\left[\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1},\widehat{y}_1=x} \left(1-\hat{p}^1\left((q;\widehat{\mathbf{y}})\right)-\hat{p}^{k+1}\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \\ &+ \sum_{k=2}^{N-1}\left[\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq q>\widehat{y}_{k+1},\widehat{y}_2=x} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \geq 0.\end{aligned}$$

Finally, if $x < r$, then $\Pi_{22}(q|x) = 0$: the buyer will never win the second auction, and his chance of getting the first item depends on his report but not his true type. Thus, $\Pi(q|x)$ is convex in the buyer's valuation, as desired.

A.3 Incentive compatibility

Truthful reporting is a best response if and only if for all $x, q \in [\underline{x}, \bar{x}]$,

$$U(x) = \Pi(x|x) - t(x) \geq (q|x) - t(q) = U(q) + \Pi(q|x) - \Pi(q|q) = U(q) + \int\limits_q^x \Pi_2(q|x')dx'. \tag{7}$$

By substituting (1) into (7), we can rewrite the incentive compatibility condition as

$$\int_q^x \Pi_2(x'|x')dx' \geq \int_q^x \Pi_2(q|x')dx'.$$

That condition holds if for any type x and any reports q, q' such that $q > x > q'$, we have $\Pi_2(q|x) \geq \Pi_2(x|x) \geq \Pi_2(q'|x)$. The allocation rules in Theorems 1, 2, and 3 have the property that $\hat{p}^k(\widehat{\mathbf{x}}) = 0$ when $k > 2$ for all $\widehat{\mathbf{x}}$, so the expressions for $\Pi_2(q|x) - \Pi_2(x|x)$ simplify. If $x \geq r$ and $q > x$, then

$$\begin{aligned} &\Pi_2(q|x) - \Pi_2(x|x) \\ = &\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1 \geq x > \widehat{y}_2} \left(\hat{p}^1((q;\widehat{\mathbf{y}})) + \hat{p}^2((q;\widehat{\mathbf{y}})) - \hat{p}^1((x;\widehat{\mathbf{y}})) - \hat{p}^2((x;\widehat{\mathbf{y}}))\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ + &\int\limits_{[\underline{x},\bar{x}]^{N-1}:q > \widehat{y}_1 > \widehat{y}_2 \geq x} \left(\hat{p}^1((q;\widehat{\mathbf{y}}))\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) + \int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1 \geq q > \widehat{y}_2 \geq x} \left(\hat{p}^2((q;\widehat{\mathbf{y}}))\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}). \end{aligned} \tag{8}$$

If $x \geq r$ and $q < x$, then

$$\begin{aligned} &\Pi_2(x|x) - \Pi_2(q|x) \\ = &\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1 \geq x > q > \widehat{y}_2} \left(\hat{p}^1((x;\widehat{\mathbf{y}})) + \hat{p}^2((x;\widehat{\mathbf{y}})) - \hat{p}^1((q;\widehat{\mathbf{y}})) - \hat{p}^2((q;\widehat{\mathbf{y}}))\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ + &\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1 \geq x > \widehat{y}_2 \geq q} \left(\hat{p}^1((x;\widehat{\mathbf{y}})) + \hat{p}^2((x;\widehat{\mathbf{y}})) - \hat{p}^1((q;\widehat{\mathbf{y}}))\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}). \end{aligned} \tag{9}$$

Because the allocation rules in Theorems 1, 2, and 3 have the property that $\hat{p}^1(\widehat{\mathbf{x}}) + \hat{p}^2(\widehat{\mathbf{x}})$ is weakly increasing in $\widehat{x}_1$ and $\widehat{x}_2$, (8) and (9) are positive, as desired.

Next consider the case $x < r$. The allocation rules in Theorems 1, 2, and 3 have the additional property that $\hat{p}^2(\widehat{\mathbf{x}}) = 0$ when $\widehat{x}_2 < r$, so for $q > x$ we have

$$\begin{aligned} & \Pi_2(q|x) - \Pi_2(x|x) \\ = & \int_{[\underline{x},\bar{x}]^{N-1}:x>\widehat{y}_1} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right) - \hat{p}^1\left((x;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ + & \int_{[\underline{x},\bar{x}]^{N-1}:q>\widehat{y}_1\geq x} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) + \int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1\geq q} \left(\hat{p}^2\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}). \end{aligned} \tag{10}$$

Finally, if $x < r$ and $q < x$, then

$$\begin{aligned} & \Pi_2(x|x) - \Pi_2(q|x) \\ = & \int_{[\underline{x},\bar{x}]^{N-1}:q>\widehat{y}_1} \left(\hat{p}^1\left((x;\widehat{\mathbf{y}})\right) - \hat{p}^1\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) + \int_{[\underline{x},\bar{x}]^{N-1}:x>\widehat{y}_1\geq q} \left(\hat{p}^1\left((x;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}). \end{aligned} \tag{11}$$

Because the specified allocation rules have the property that $\hat{p}^1(\widehat{\mathbf{x}})$ is weakly increasing in $\widehat{x}_1$, (10) and (11) are both positive as well. We conclude that the mechanisms in Theorems 1, 2, and 3 are incentive compatible. It remains only to show that the allocation rules solve the seller's revenue maximization problem. We made that argument for the $r \in [\psi^{-1}(0), \bar{x})$ case in Section 3.1. We cover the other cases next.

A.4 Constrained Optimization when $r \in \left(\underline{x}, \psi^{-1}(0)\right)$

Recall that the seller's problem is to maximize (2) subject to $\int_q^r \left[\Pi_2(x'|x') - \Pi_2(q|x')\right] dx' \geq 0$ for each $q \in [\underline{x}, r)$ (with Lagrange multiplier $\lambda_{r,q}$), and to

$$\Pi_2(x|x) - \limsup_{\epsilon \searrow 0} \Pi_2(x-\epsilon|x) \geq 0$$

for each $x \in (\underline{x}, a(r)]$ (multiplier μ_x). Using the derivations in Section A.1.1, we write out

$$\begin{aligned} & \Pi_2(x|x) - \limsup\nolimits_{\epsilon\searrow 0} \Pi_2(x-\epsilon|x) = \\ & \limsup\nolimits_{\epsilon\searrow 0} \left\{ \begin{array}{cc} & \int_{[\underline{x},\bar{x}]^{N-1}:x>\widehat{y}_1} \left[\hat{p}^1\left((x;\widehat{\mathbf{y}})\right) - \hat{p}^1\left((x-\epsilon;\widehat{\mathbf{y}})\right)\right]\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ + & \sum_{k=1}^{N-1} \left[\int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k\geq x>\widehat{y}_{k+1}} \left[\hat{p}^{k+1}\left((x;\widehat{\mathbf{y}})\right) - \hat{p}^{k+1}\left((x-\epsilon;\widehat{\mathbf{y}})\right)\right]\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \end{array} \right\} \end{aligned}$$

for each $x \in [\underline{x}, r)$;

$$\Pi_2(x|x) - \limsup_{\epsilon \searrow 0} \Pi_2(x-\epsilon|x) =$$

$$\limsup_{\epsilon\searrow 0}\left\{\begin{array}{c} \int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1 \geq x > \widehat{y}_2} \left[\begin{array}{c} \left(\hat{p}^1\left((x;\widehat{\mathbf{y}})\right) - \hat{p}^1\left((x-\epsilon;\widehat{\mathbf{y}})\right)\right) \\ +\left(\hat{p}^2\left((x;\widehat{\mathbf{y}})\right) - \hat{p}^2\left((x-\epsilon;\widehat{\mathbf{y}})\right)\right) \end{array}\right] \widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ + \quad \sum_{k=2}^{N-1}\left[\int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k \geq x > \widehat{y}_{k+1}} \left(\hat{p}^{k+1}\left((x;\widehat{\mathbf{y}})\right) - \hat{p}^{k+1}\left((x-\epsilon;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \end{array}\right\}$$

for each $x \in [r, a(r)]$;

$$\begin{array}{c} \Pi_2(r|r) - \Pi_2(q|r) = \\ \int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1 \geq r > \widehat{y}_2} \left(\hat{p}^1\left((r;\widehat{\mathbf{y}})\right) + \hat{p}^2\left((r;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ + \quad \sum_{k=2}^{N-1}\left[\int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k \geq r > \widehat{y}_{k+1}} \hat{p}^{k+1}\left((r;\widehat{\mathbf{y}})\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \\ - \left[\begin{array}{c} \int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1 \geq r, \widehat{y}_2 < q} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right) + \hat{p}^2\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ + \quad \sum_{k=1}^{N-1}\left[\int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_1 \geq r, \widehat{y}_2 < r, \widehat{y}_k \geq q > \widehat{y}_{k+1}} \left(\begin{array}{c} \hat{p}^1\left((q;\widehat{\mathbf{y}})\right) \\ +\hat{p}^{k+1}\left((q;\widehat{\mathbf{y}})\right) \end{array}\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \\ + \quad \sum_{k=2}^{N-1}\left[\int_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_2 \geq r, \widehat{y}_k \geq q > \widehat{y}_{k+1}} \hat{p}^{k+1}\left((q;\widehat{\mathbf{y}})\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \end{array}\right] \end{array}$$

for each $q \in [\underline{x}, r)$; and

$$\begin{array}{c} \Pi_2(x|x) - \Pi_2(q|x) = \\ \int\limits_{[\underline{x},\bar{x}]^{N-1}:x > \widehat{y}_1} \left(\hat{p}^1\left((x;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ + \sum_{k=1}^{N-1}\left[\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k \geq x > \widehat{y}_{k+1}} \hat{p}^{k+1}\left((x;\widehat{\mathbf{y}})\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \\ - \left[\begin{array}{c} \int\limits_{[\underline{x},\bar{x}]^{N-1}:q > \widehat{y}_1} \left(\hat{p}^1\left((q;\widehat{\mathbf{y}})\right)\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}}) \\ + \sum_{k=1}^{N-1}\left[\int\limits_{[\underline{x},\bar{x}]^{N-1}:\widehat{y}_k \geq q > \widehat{y}_{k+1}} \hat{p}^{k+1}\left((q;\widehat{\mathbf{y}})\right)\widehat{\mathbf{g}}(\widehat{\mathbf{y}})\right] \end{array}\right] \end{array}$$

for each $x \in [\underline{x}, r)$ and $q \in [\underline{x}, x)$. Note that for $x \geq r$, the integrals do not include the case $x > \widehat{y}_1$, because the highest-type buyer is sure to get an object when his type is above r.

A.4.1 When $Z^r(r) \geq 0$

In this case, Theorem 3 specifies allocation whenever either i) $\widehat{x}_1 \geq r > \widehat{x}_3$, or ii) $\widehat{x}_3 \geq r$ and $\psi(\widehat{x}_2) + \widehat{x}_2 - \widehat{x}_3 \geq 0$, and not otherwise. We make the following guesses for the values of the Lagrange multipliers: for all $x \in [\underline{x}, r)$, $\lambda_{r,x} = \mu_x = 0$; and for $x \in [r, a(r)]$, $\mu_x = \int_x^{a(r)} N \cdot [r - x' - \psi(x')] f(x') dx'$. That is, only the immediate downward constraints for type r and above bind. The intuition behind that guess for the values of μ_x is as follows: suppose that we relaxed the constraint that type $x \in [r, a(r)]$ must have a weakly higher chance of getting an object if he reports truthfully than if he underreports to $x - \epsilon$. Then the seller could not allocate when $\widehat{x}_2 = x$ and $\widehat{x}_3 < r$, and thus avoid earning the negative marginal revenue $x + \psi(x) - r$ in that case. There is an additional benefit: for type $x + \epsilon$, now underreporting does not lead to allocation, and so the seller is free to not allocate when $\widehat{x}_2 = x + \epsilon$ and $\widehat{x}_3 < r$, without violating the constraint for type $x + \epsilon$. Iterating, we see that relaxing the constraint for the *single* type x allows the seller to avoid the negative marginal revenue $x' + \psi(x') - r$ for *every* type x' between x and $a(r)$.

In what follows, the key feature of μ_x is that $\mu_x - \mu_{x+0} = N \cdot [r - x - \psi(x)] f(x) dx$. For example, suppose that $\widehat{x}_2 \in [r, a(r))$. Allocating to either of the top two bidders in that case helps with the constraint for type $\widehat{x}_2$ (he gets an item by telling the truth), but it hurts with the constraint for a slightly higher type (he could get an item by underreporting his type as $\widehat{x}_2$). The net marginal effect is the difference between $\mu_{\widehat{x}_2}$ and $\mu_{\widehat{x}_2+0}$.

We use that key feature repeatedly as we next take the partial derivative of the seller's expected revenue with respect to $\hat{p}^k(\hat{\mathbf{x}})$, given any vector of ordered types $\hat{\mathbf{x}}$, and plug in those guesses. Note that for any $\widehat{\mathbf{x}}$ and any k, $N \cdot f(\widehat{x}_k) \cdot \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k}) = \hat{\mathbf{f}}(\widehat{\mathbf{x}})$.

1. If $\widehat{x}_2 \geq a(r)$, then $\frac{\partial ER(\widehat{X})}{\delta \hat{p}^1(\hat{\mathbf{x}})} = \frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})} = [\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\}] \hat{\mathbf{f}}(\widehat{\mathbf{x}})$.

2. If $\widehat{x}_2 \in [r, a(r))$, then

$$\begin{aligned}\frac{\partial ER(\widehat{X})}{\delta \hat{p}^1(\hat{\mathbf{x}})} = \frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})} &= [\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\}]\, \hat{\mathbf{f}}(\widehat{\mathbf{x}}) + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\mu_{\widehat{x}_2} - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\mu_{\widehat{x}_2+0} \\ &= [\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\}]\, \hat{\mathbf{f}}(\widehat{\mathbf{x}}) \\ &+ \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})N \cdot [r - \widehat{x}_2 - \psi(\widehat{x}_2)]\, f(\widehat{x}_2) \\ &= \hat{\mathbf{f}}(\widehat{\mathbf{x}}) \cdot [\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\} + (r - \widehat{x}_2 - \psi(\widehat{x}_2))] \\ &= \hat{\mathbf{f}}(\widehat{\mathbf{x}}) \cdot [r - \max\{\widehat{x}_3, r\}]\,.\end{aligned}$$

3. If $\widehat{x}_1 \geq r$ and $\widehat{x}_2 < r$, then $\frac{\partial ER(\widehat{X})}{\delta \hat{p}^1(\hat{\mathbf{x}})} = r\hat{\mathbf{f}}(\widehat{\mathbf{x}}) > 0$.

4. In every case above, for all $k > 2$ such that $\widehat{x}_k \geq a(r)$,

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} = \psi(\widehat{x}_k)\, \hat{\mathbf{f}}(\widehat{\mathbf{x}}) \leq \frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})};$$

for all $k > 2$ such that $\widehat{x}_k \in [r, a(r))$,

$$\begin{aligned}\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} &= \psi(\widehat{x}_k)\, \hat{\mathbf{f}}(\widehat{\mathbf{x}}) + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\mu_{\widehat{x}_k} - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\mu_{\widehat{x}_k+0} \\ &= \psi(\widehat{x}_k)\, \hat{\mathbf{f}}(\widehat{\mathbf{x}}) + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})N \cdot [r - \widehat{x}_k - \psi(\widehat{x}_k)]\, f(\widehat{x}_k) \\ &= \hat{\mathbf{f}}(\widehat{\mathbf{x}}) \cdot [\psi(\widehat{x}_k) + (r - \widehat{x}_k - \psi(\widehat{x}_k))] = \hat{\mathbf{f}}(\widehat{\mathbf{x}}) \cdot [r - \widehat{x}_k] < 0;\end{aligned}$$

and for all k such that $\widehat{x}_k < r$, $\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} = \psi(\widehat{x}_k)\, \hat{\mathbf{f}}(\widehat{\mathbf{x}}) < 0$.

The marginal revenues above are weakly positive in each case where Theorem 3 specifies allocation, and they are weakly negative in each case where Theorem 3 specifies no allocation. Thus, our guesses for the values of the Lagrange multipliers, together with the allocation rule in Theorem 3, form a solution to the seller's constrained optimization problem.

A.4.2 When $Z^r(r) \leq 0$

In this case, Theorem 2 specifies allocation if and only if $\psi(\widehat{x}_2)+\widehat{x}_2-\max\{r,\widehat{x}_3\}\geq 0$. The derivative of $Z^r(x^*)$ is given by $z^r(x^*)\,f(x^*)$, where the function $z^r(x^*)$ is defined as

$$z^r(x^*) \equiv -rF(r) + (N-1)\int_r^{\min\{x^*,a(r)\}} [r - x' - \psi(x')]f(x')dx'.$$

The function $z^r(x^*)$ is strictly increasing for $x^* < a(r)$ and constant for $x^* \geq a(r)$. When $Z^r(\bar{x}) - Z^r(r) = 0 - Z^r(r) \geq 0$, therefore, it must be that $z^r(a(r)) \geq 0$: otherwise $Z^r(x^*)$ would be strictly decreasing throughout. We will use the inequality $z^r(a(r)) \geq 0$ below.

We make the following guesses for the Lagrange multipliers: for $x \in [\underline{x}, r)$, $\lambda_{r,x} = r\frac{N}{N-1}f(x)$ and $\mu_x = \int_{\underline{x}}^{x} r\frac{N}{N-1}F(x')dx'$; and for $x \in (r, a(r)]$, $\mu_x = \int_x^{a(r)} N\cdot[r - x' - \psi(x')]\,f(x'$ The differences relative to the $Z^r(r) \geq 0$ case are that now all the downward constraints bind for type r, and the immediate downward constraints bind for types below r. Allocating to either of the top two bidders when $\widehat{x}_2 = r$ helps with all the downward constraints for type r, because he gets an item by telling the truth. On the other hand, allocating to the highest bidder when $\widehat{x}_1 \geq r > \widehat{x}_2$ hurts with a constraint, because a bidder with type r gets an object by underreporting his type as $\widehat{x}_2$. The intuition for our guess of the value of $\lambda_{r,x}$ is that if we relaxed the constraint, then the seller could allocate to the high bidder whenever $\widehat{x}_1 \geq r > \widehat{x}_2$ and earn the corresponding marginal revenue r.

As we take the partial derivative of the seller's expected revenue with respect to $\hat{p}^k(\hat{\mathbf{x}})$ and plug in those guesses, we again use the feature that $\mu_x - \mu_{x+0} = N\cdot[r - x - \psi(x)]\,f(x)dx$ for $x \in (r, a(r)]$. Similarly, we use the feature that for $x \in [\underline{x}, r)$, $\mu_{x+0} - \mu_x = r\frac{N}{N-1}F(x)dx$.

1. If $\widehat{x}_2 \geq a(r)$, then

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^1(\hat{\mathbf{x}})} = [\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\}]\,\hat{\mathbf{f}}(\widehat{\mathbf{x}}) - \sum_{k:\widehat{x}_k < r} [\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\lambda_{r,\widehat{x}_k}] \leq \frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})};$$

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})} = \left[\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\}\right] \hat{\mathbf{f}}(\widehat{\mathbf{x}}).$$

2. If $\widehat{x}_2 \in (r, a(r))$, then

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^1(\hat{\mathbf{x}})} = \begin{pmatrix} \left[\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\}\right] \hat{\mathbf{f}}(\widehat{\mathbf{x}}) + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\mu_{\widehat{x}_2} \\ -\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\mu_{\widehat{x}_2+0} - \sum_{k:\widehat{x}_k<r} \left[\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\lambda_{r,\widehat{x}_k}\right] \end{pmatrix} \leq \frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})};$$

$$\begin{aligned}
\frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})} &= \left[\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\}\right] \hat{\mathbf{f}}(\widehat{\mathbf{x}}) + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\mu_{\widehat{x}_2} - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\mu_{\widehat{x}_2+0} \\
&= \left[\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\}\right] \hat{\mathbf{f}}(\widehat{\mathbf{x}}) + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})N \cdot \left[r - \widehat{x}_2 - \psi(\widehat{x}_2)\right] f(\widehat{x}_2) \\
&= \hat{\mathbf{f}}(\widehat{\mathbf{x}}) \cdot \left[\widehat{x}_2 + \psi(\widehat{x}_2) - \max\{\widehat{x}_3, r\} + (r - \widehat{x}_2 - \psi(\widehat{x}_2))\right] \\
&= \hat{\mathbf{f}}(\widehat{\mathbf{x}}) \cdot \left[r - \max\{\widehat{x}_3, r\}\right].
\end{aligned}$$

3. If $\widehat{x}_2 = r$, then

$$\begin{aligned}
\frac{\partial ER(\widehat{X})}{\delta \hat{p}^1(\hat{\mathbf{x}})} &= \left[r + \psi(r) - r\right] \hat{\mathbf{f}}(\widehat{\mathbf{x}}) - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\mu_{r+0} + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2}) \int_{\underline{x}}^{r} \lambda_{r,x'} dx' - \sum_{k=3}^{N} \left[\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\lambda_{r,\widehat{x}_k}\right] \\
&\leq \frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})};
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})} &= [r+\psi(r)-r]\,\hat{\mathbf{f}}(\widehat{\mathbf{x}}) - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\mu_{r+0} + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\int_{\underline{x}}^{r} \lambda_{r,x'}dx' \\
&= [r+\psi(r)-r]\,\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})Nf(r) - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\int_{r+0}^{a(r)} N\cdot[r-x-\psi(x)]\,f(x)dx \\
&+ \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})F(r)\frac{N}{N-1}r \\
&= -\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\int_{r}^{a(r)} N\cdot[r-x-\psi(x)]\,f(x)dx + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})F(r)\frac{N}{N-1}r \\
&= \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\frac{N}{N-1}\left[F(r)r-(N-1)\int_{r}^{a(r)}[r-x-\psi(x)]\,f(x)dx\right] \\
&= -\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-2})\frac{N}{N-1}z^r(a(r)) \leq 0.
\end{aligned}
$$

4. If $\widehat{x}_1 \geq r$ and $\widehat{x}_2 < r$, then

$$
\frac{\partial ER(\widehat{X})}{\delta \hat{p}^1(\hat{\mathbf{x}})} = r\hat{\mathbf{f}}(\widehat{\mathbf{x}}) - \sum_{k=2}^{N}[\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\lambda_{r,\widehat{x}_k}] = r\hat{\mathbf{f}}(\widehat{\mathbf{x}}) - \sum_{k=2}^{N}\left[\hat{\mathbf{f}}(\widehat{\mathbf{x}})\frac{1}{N-1}r\right] = \hat{\mathbf{f}}(\widehat{\mathbf{x}})\cdot[r-r] = 0.
$$

5. In every case above, for all $k > 2$ such that $\widehat{x}_k \geq a(r)$,

$$
\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} = \psi(\widehat{x}_k)\,\hat{\mathbf{f}}(\widehat{\mathbf{x}}) \leq \frac{\partial ER(\widehat{X})}{\delta \hat{p}^2(\hat{\mathbf{x}})};
$$

for all $k > 2$ such that $\widehat{x}_k \in (r, a(r)]$,

$$
\begin{aligned}
\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} &= \psi(\widehat{x}_k)\,\hat{\mathbf{f}}(\widehat{\mathbf{x}}) + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\mu_{\widehat{x}_k} - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\mu_{\widehat{x}_k+0} \\
&= \psi(\widehat{x}_k)\,\hat{\mathbf{f}}(\widehat{\mathbf{x}}) + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})N\cdot[r-\widehat{x}_k-\psi(\widehat{x}_k)]\,f(\widehat{x}_k) \\
&= \hat{\mathbf{f}}(\widehat{\mathbf{x}})\cdot[\psi(\widehat{x}_k)+(r-\widehat{x}_k-\psi(\widehat{x}_k))] = \hat{\mathbf{f}}(\widehat{\mathbf{x}})\cdot[r-\widehat{x}_k] < 0;
\end{aligned}
$$

for all $k > 2$ such that $\widehat{x}_k = r$,

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} = [\psi(r)]\,\hat{\mathbf{f}}(\widehat{\mathbf{x}}) - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\mu_{r+0} + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\int_{\underline{x}}^{r} \lambda_{r,x'}dx'$$
$$= [r + \psi(r) - r]\,\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})Nf(r) - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\int_{r+0}^{a(r)} N\cdot[r - x - \psi(x)]\,f(x)dx$$
$$+ \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})F(r)\frac{N}{N-1}r$$
$$= -\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\int_{r}^{a(r)} N\cdot[r - x - \psi(x)]\,f(x)dx + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})F(r)\frac{N}{N-1}r$$
$$= \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\frac{N}{N-1}\left[F(r)r - (N-1)\int_{r}^{a(r)} N\cdot[r - x - \psi(x)]\,f(x)dx\right]$$
$$= -\hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\frac{N}{N-1}z^r(a(r)) \leq 0;$$

and for all k such that $\widehat{x}_k < r$,

$$\frac{\partial ER(\widehat{X})}{\delta \hat{p}^k(\hat{\mathbf{x}})} = \psi(\widehat{x}_k)\,\hat{\mathbf{f}}(\widehat{\mathbf{x}}) - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\mu_{\widehat{x}_k+0} + \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\mu_{\widehat{x}_k}$$
$$+ \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\int_{\underline{x}}^{\widehat{x}_k} \lambda_{r,x'}dx' - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\int_{\widehat{x}_k}^{r} \lambda_{r,\widehat{x}_k}dx'$$
$$= \psi(\widehat{x}_k)\,\hat{\mathbf{f}}(\widehat{\mathbf{x}}) - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\left[r\frac{N}{N-1}F(\widehat{x}_k)\right]$$
$$+ \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})F(\widehat{x}_k)\frac{N}{N-1}r - \hat{\mathbf{g}}(\widehat{\mathbf{x}}_{-k})\int_{\widehat{x}_k}^{r} \lambda_{r,\widehat{x}_k}dx' \leq \psi(\widehat{x}_k)\,\hat{\mathbf{f}}(\widehat{\mathbf{x}}) < 0.$$

The marginal revenues above are weakly positive in each case where Theorem 2 specifies allocation, and they are weakly negative in each case where Theorem 2 specifies no allocation. Thus, our guesses for the values of the Lagrange multipliers, together with the allocation rule in Theorem 2, form a solution to the seller's constrained optimization problem.

B Proving Theorem 5

Theorems 1, 2, and 3 show that seller 1 has a best response to any r that seller 2 chooses. To establish existence of an equilibrium, then, we need only show that there exists a maximizer of $R_2(r)$, seller 2's revenue when she sets reserve price r and seller 1 best responds. (In case seller 1 has multiple best responses, let her choose one that maximizes seller 2's revenue.) A maximizer exists because $R_2(r)$ is upper semicontinuous: first, note that seller 2's revenue is continuous in r and in seller 1's allocation rule. For $r > \psi^{-1}(0)$, seller 1's allocation rule is constant (Theorem 1). For $r < \psi^{-1}(0)$, Theorems 2 and 3 show that the sets of values of r where cutoffs $x^* = r$ and $x^* = \bar{x}$, respectively, are optimal for seller 1 are closed (because $Z^r(r)$ is continuous), and that within each set seller 1's allocation rule is continuous in r. Finally, seller 1's allocation rule is continuous in r at $r = \psi^{-1}(0)$:

$$\lim_{r \nearrow \psi^{-1}(0)} Z^r(r) = \lim_{r \nearrow \psi^{-1}(0)} rF(r)\left[1 - F(r)\right] > 0,$$

and Theorem 3 then implies that for r just below $\psi^{-1}(0)$, $x^* = r$.

Thus, $R_2(r)$ is upper semicontinuous, and so it has a maximizer r^* on the compact set $[\underline{x}, \bar{x}]$. Because $R_2(r) = R_2(\underline{x})$ for all $x < \underline{x}$, and $R_2(r) = R_2(\bar{x})$ for all $r > \bar{x}$, r^* is the global maximizer of $R_2(r)$.

Next, we show that $r^* < \psi^{-1}(0)$ by establishing that $R_2(r)$ is decreasing in r at $r = \psi^{-1}(0)$ and above. For $r \geq \psi^{-1}(0)$, Theorem 1 implies that seller 2's revenue is

$$\int_r^{\bar{x}} \left[rF_{3|x_{(2)}}(r) + \int_r^{x_{(2)}} \widehat{x}_3 f_{3|x_{(2)}}(\widehat{x}_3) d\widehat{x}_3 \right] f_2(\widehat{x}_2) d\widehat{x}_2.$$

The derivative with respect to r is

$$\begin{array}{rc} & -rf_2(r) + \int_r^{\bar{x}} \left[F_{3|x_{(2)}}(r) f_2(\widehat{x}_2) d\widehat{x}_2 \right] \\ = & N(N-1)\left(1-F(r)\right)\left[F(r)\right]^{N-2} f(r) \left\{ -r + \int_r^{\bar{x}} \left[\left(\frac{1-F(\widehat{x}_2)}{1-F(r)} \right) \frac{f(\widehat{x}_2)}{f(r)} d\widehat{x}_2 \right] \right\} \\ < & N(N-1)\left(1-F(r)\right)\left[F(r)\right]^{N-2} f(r) \left\{ -r + \int_r^{\bar{x}} \frac{f(\widehat{x}_2)}{f(r)} d\widehat{x}_2 \right\} \\ = & -N(N-1)\left(1-F(r)\right)\left[F(r)\right]^{N-2} \left\{ r - \frac{1-F(r)}{f(r)} \right\} \\ = & -N(N-1)\left(1-F(r)\right)\left[F(r)\right]^{N-2} \psi(r) \leq 0. \end{array}$$

The last inequality follows from the assumption that ψ is increasing.

As noted above, $x^* = r$ for r just below $\psi^{-1}(0)$, and so seller 2's revenue in that case is

$$\begin{array}{rc} & \int_r^{a(r)} \left[rF_{3|x_{(2)}}(r) + \widehat{x}_2 \left[1 - F_{3|x_{(2)}}(r) \right] \right] f_2(\widehat{x}_2) d\widehat{x}_2 \\ + & \int_{a(r)}^{\psi^{-1}(0)} \left[\begin{array}{cc} & rF_{3|x_{(2)}}(r) + \widehat{x}_2 \left[1 - F_{3|x_{(2)}}(\widehat{x}_2 + \psi(\widehat{x}_2)) \right] \\ + & \int_r^{\widehat{x}_2 + \psi(\widehat{x}_2)} \widehat{x}_3 f_{3|x_{(2)}}(\widehat{x}_3) d\widehat{x}_3 \end{array} \right] f_2(\widehat{x}_2) d\widehat{x}_2 \\ + & \int_{\psi^{-1}(0)}^{\bar{x}} \left[rF_{3|x_{(2)}}(r) + \int_r^{x(2)} \widehat{x}_3 f_{3|x_{(2)}}(\widehat{x}_3) d\widehat{x}_3 \right] f_2(\widehat{x}_2) d\widehat{x}_2. \end{array}$$

The derivative with respect to r evaluated at $r = \psi^{-1}(0)$ (the value at which $a(r) = r$) is

$$\begin{array}{rc} & -\psi^{-1}(0) f_2(\psi^{-1}(0)) + \int_{\psi^{-1}(0)}^{\bar{x}} \left[F_{3|x_{(2)}}(\psi^{-1}(0)) f_2(\widehat{x}_2) d\widehat{x}_2 \right] \\ < & -N(N-1)\left(1-F(\psi^{-1}(0))\right)\left[F(\psi^{-1}(0))\right]^{N-2} \psi(\psi^{-1}(0)) = 0. \end{array}$$

Thus, $R_2(r)$ is strictly decreasing in r for $r \geq \psi^{-1}(0)$, so $r^* < \psi^{-1}(0)$.

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